Pergamon

J. Appl. Maths Mechs, Vol. 62, No. 4, pp. 631–634, 1998

0021–8928/98/$—see front matter
PII: S0021–8928(98)00079-3

THE WELL-POSEDNESS OF THE STATICS PROBLEM OF THE NON-LINEAR THEORY OF SHALLOW ELASTIC SHELLS†

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(*Received* 21 *January* 1998)

The continuity of the dependence of the non-singular solution on small perturbations of the dimensions and form of the shell is proved using methods described earlier [1]. These perturbations lead to a change in the region into which the middle surface of the shell is mapped (for example, an increase or decrease in the aperture angle of a shallow spherical cupola). The continuity of the dependence on small changes in parts of the boundary along which some form of boundary conditions is realized (for example, there is some part of the boundary rigidly clamped with respect to the displacement of points in the direction of normal to the middle surface) is also proved. © 1998 Elsevier Science Ltd. All rights reserved.

The continuity of the dependence of non-singular solutions on small changes in the elastic characteristics of a shell and on small perturbations of the form of the shell, which do not give rise to any changes in the region in which the coordinates of the middle surface and parts of the boundaries with different types of boundary conditions are specified, was proved earlier [1].

We will consider the general boundary-value problem of the non-linear theory of elastic shells of average curvature within the framework of the Kirchhoff–Love hypothesis. It is assumed that the shell has a fairly smooth middle surface S, mapped into a connected bounded open set Ω with a piecewise-smooth boundary $\partial\Omega$ in the plane R^2. The curvilinear coordinates on the middle surface $\boldsymbol{\xi} = (\xi^1, \xi^2) \in \Omega$ define the vectors of the fundamental basis $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\xi^\alpha$, where $\mathbf{r} = \mathbf{r}(\xi^1, \xi^2)$ is the equation of the non-deformed middle surface of the shell. Together with the vector of the normal $\mathbf{n} = \mathbf{a}_3 = \mathbf{a}^3$ to the middle surface of the vector of the main basis they form a three-dimensional basis which varies along S. The mutual basis $\mathbf{a}^\alpha$ is defined by the relations $\mathbf{a}^\alpha\cdot\mathbf{a}_\beta = \delta^\alpha_\beta$, where δ^α_β is the Kronecker delta. Below we will use the rule for summation over repeated subscripts and superscripts.

In the theory of shallow shells the "mean" bending deformations are described by two tensors, the strain tensor of the middle surface $\boldsymbol{\gamma} = \gamma_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta$ and the tensor of the change in the curvature of the middle surface $\boldsymbol{\rho} = \rho_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta$, the components of which have the form

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \tfrac{1}{2}\varphi_\alpha\varphi_\beta, \quad \rho_{\alpha\beta} = \tfrac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}), \quad \theta_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w, \quad \varphi_\alpha = w_{,\alpha}$$

Here $b_{\alpha\beta} = b_{\beta\alpha} = -\mathbf{a}_\alpha\cdot\mathbf{a}_{3,\beta}$ are the coefficients of the second quadratic form of the middle surface, the subscript β after the comma denotes a partial derivative with respect to ξ^β, while $\Gamma^\alpha{}_{\lambda\beta}$ are the Christoffel symbols. In addition, we will introduce the following notation, written for the coordinates of the displacement vector $\mathbf{u} = u_i\mathbf{a}^i = u^i\mathbf{a}_i$ ($w \equiv u^3 \equiv u_3$)

$$u_{\alpha|\beta} = u_{\alpha,\beta} - \Gamma^\varkappa_{\alpha\beta}u_\varkappa, \quad u^\alpha_{|\beta} = u^\alpha_{,\beta} + \Gamma^\alpha_{\beta\varkappa}u^\varkappa, \quad u_{3|\alpha} = u_{3,\alpha}$$

In the case when a quantity is in terms of the displacement vector $\mathbf{u}$, expressed in terms of the vector $\mathbf{v} = \upsilon^i\mathbf{a}_i$, rather than this is denoted by the notation in brackets. For example, $\varphi_\alpha(\mathbf{v}) = \upsilon_{3,\alpha}$. The relations of the theory of shallow shells can be found in [1, 2].

As a consequence of the linear distribution of the displacements over the thickness of the shell, the stress tensor can be split into two components, one of which represents the longitudinal forces in the shell $\mathbf{n} = n_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta$, while the other represents the moments $\mathbf{m} = m_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta$. The equations of equilibrium of the shell, written using the virtual work principle, have the form

†*Prikl. Mat. Mekh.* Vol. 62, No. 4, pp. 678–682, 1998.

$$\int_\Omega (n^{\alpha\beta}\delta\gamma_{\alpha\beta}+m^{\alpha\beta}\delta\rho_{\alpha\beta})d\Omega-\int_\Omega (F^1\delta u_1+F^2\delta u_2+F^3\delta w)d\Omega-$$
$$-\int_{\partial\Omega}(f^3\delta w+M^*\partial\delta w/\partial n)ds=0,\quad d\Omega=Jd\xi^1 d\xi^2,\quad J^2=a_{11}a_{22}-a_{12}^2\geqslant c_0>0 \tag{1}$$

where $a_{\alpha\beta}=a_{\beta\alpha}=\mathbf{a}_\alpha\cdot\mathbf{a}_\beta$ are the coefficients of the first quadratic form of the middle surface, ρ is the density of the material, h is the shell thickness and F^i are the external surface forces. The coordinates of the vector of virtual displacements δu_i are related to the variations in the strain tensors as follows:

$$\delta\gamma_{\alpha\beta}=\tfrac{1}{2}(\delta u_{\alpha|\beta}+\delta u_{\beta|\alpha})-b_{\alpha\beta}\delta w+\tfrac{1}{2}(\varphi_\beta\delta\varphi_\alpha+\varphi_\alpha\delta\varphi_\beta)$$
$$\delta\rho_{\alpha\beta}=\tfrac{1}{2}(\delta\varphi_{\alpha|\beta}+\delta\varphi_{\beta|\alpha}),\quad \delta\varphi_\alpha=\delta w_{,\alpha}+b_{\alpha\lambda}\delta u^\lambda=\delta w_{,\alpha}+b_\alpha^\lambda\delta u_\lambda$$

whereas the strain tensors are related to the stress tensors and the moments as follows:

$$n^{ij}=hc^{ijkl}\gamma_{kl},\quad m^{ij}=h^3c^{ijkl}\rho_{kl}/12$$

where c^{ijkl} is the elasticity constants tensor of the material, which possess the properties of symmetry and positive definiteness.

Using standard methods of the variational calculus we can obtain from Eq. (1) the differential equations of equilibrium

$$n^{\beta\alpha}_{|\beta}+F^\alpha=0,\quad m^{\alpha\beta}_{|\alpha\beta}-b_{\alpha\beta}n^{\alpha\beta}-(u_{3|\alpha}n^{\alpha\beta})_{|\beta}-F^3=0,\quad \alpha=1,2$$

and the natural boundary conditions, which can be found in [1, 2] in a different notation.

For simplicity we will consider the following version of the boundary conditions

$$u_1|_\Gamma=u_2|_\Gamma=0,\quad w|_{\Gamma_1}=0,\quad \partial w/\partial n|_{\Gamma_1}=0 \tag{2}$$

where Γ_1 is a certain part of the boundary contour Γ. On the remaining part of the boundary we will use the natural boundary conditions.

We will introduce the following scalar product on the set C of vector function $\mathbf{u}(\xi)$, $\delta\mathbf{u}(\xi)\in C^{(2)}(\Omega)$, which satisfy boundary conditions (2),

$$(\mathbf{u},\delta\mathbf{u})_H=\int_\Omega\left(hc^{ijkl}\theta_{kl}\delta\theta_{ij}+\frac{h^3}{12}c^{ijkl}\rho_{kl}\delta\rho_{ij}\right)d\Omega$$

The closure of the set C of vector functions in the appropriate norm $\|\mathbf{u}\|_H=(\mathbf{u},\mathbf{u})_H^{1/2}$ is the energy space H.

Lemma 1. The components of the vector $\mathbf{u}\in H$ are the elements of the spaces respectively: u_1, u^1, u_2, $u^2\in W_2^{(1)}(\Omega)$, $u_3=u^3=w\in W_2^{(2)}(\Omega)$; moreover, the corresponding norms of the spaces H and $W_2^{(1)}(\Omega)\times W_2^{(1)}(\Omega)\times W_2^{(2)}(\Omega)$ are equivalent.

Definition 1. The vector function $\mathbf{u}\in H$, which satisfies Eq. (1) for any $\delta\mathbf{u}\in H$, is called the generalized solution of the problem of the equilibrium of a shallow elastic shell.

In order for this definition to be well-posed, it is necessary to impose additional conditions on the components of the vectors of the external forces. Namely, we will assume that the vectors of the external forces are such that the functional of the work of the external forces, which is linear in $\delta\mathbf{u}$

$$\int_\Omega(F^1\delta u_1+F^2\delta u_2+F^3\delta w)\,d\Omega+\int_{\partial\Omega}(f^3\delta w+M^*\partial\delta w/\partial n)ds$$

will be a continuous linear functional in H with respect to $\delta\mathbf{u}$. By Sobolev's imbedding theorems [4], the following conditions are the sufficient conditions for this: F^1, $F^2\in L^p(\Omega)$, $M^*\in L^p(\partial\Omega)$, for any finite $p>1$, $F^3=F_0^3+F_1^3$, $f^3=f_0^3+f_1^3$, $F_0^3\in L(\Omega$, while F_1^3 and f_1^3 are certain finite linear combinations of δ-functions.

With this condition, using Riesz's theorem on the representation of a continuous linear functional in Hilbert space, we can represent the functional of the work of the external forces in the form of the scalar product

$$\int_{\Omega}(F^1\delta u_1 + F^2\delta u_2 + F^3\delta w)\,d\Omega + \int_{\partial\Omega}(f^3\delta w + M^*\partial\delta w/\partial n)\,ds = (\mathbf{g}, \delta\mathbf{u})_H \tag{3}$$

Following the well known procedure [1, 3], we will change from the integro-differential equation (1) to an operator equation in the energy space H. To do this we separate the term $(\mathbf{u}, \delta\mathbf{u})_H$ on the left-hand side of Eq. (1). It has been shown [1, 3] that the remaining terms on the left-hand side, for fixed $\mathbf{u} \in H$, give a continuous linear functional in the variable $\delta\mathbf{u} \in H$, and hence, by Riesz's theorem on the representation of a continuous linear functional, their algebraic sum can be represented in the form of the scalar product $(\mathbf{G}, \delta\mathbf{u})_H$. The element $\mathbf{G}$ is defined uniquely by the element $\mathbf{u}$ and the external loads. Since the external loads are fixed, the element $\mathbf{G}$ can be regarded as the value of a certain non-linear operator at the point u, which will be denoted by $\mathbf{G} = \mathbf{G}(\mathbf{u})$. In this notation, Eq. (1) can be represented in the form $(\mathbf{u}, \delta\mathbf{u})_H = (\mathbf{G}(\mathbf{u}), \delta\mathbf{u})_H$ or, which is the same thing

$$\mathbf{u} = \mathbf{G}(\mathbf{u}) \tag{4}$$

It has been shown [1], that Eq. (4), with the above conditions imposed on the shell geometry and the external loads, has a solution and, consequently, the problem considered also has a generalized solution in the above sense.

We will later need the notion of a non-singular solution of the problem. The solution $\mathbf{u}$ of Eq. (1) is said to be singular if at the given point $\mathbf{u}$ the Frechet derivative of the operator $\mathbf{u} - \mathbf{G}(\mathbf{u})$ vanishes. The equation which governs the equality of a Frechet derivative at the point $\mathbf{u}$ to zero, if u is a singular solution, must have a non-trivial solution. A detailed investigation of this equation can be found in [1].

If a non-trivial solution of this equation does not exist, the corresponding solution $\mathbf{u}$ of Eq. (1) is said to be non-singular. At such non-singular points the Frechet derivative of the operator $\mathbf{u} - \mathbf{G}(\mathbf{u})$, being a continuous linear operator with specific properties, has a continuous inverse operator.

We will assume that there are two shallow shells (generally speaking, described by different equations), which occupy in plan the "close" regions Ω' and Ω''. All the quantities relating to the first shell will be denoted by a single prime, while for the second shell they will be denoted by a double prime. Hence, for example, part of the boundary conditions (2) takes the form

$$w|_{\Gamma_1'} = 0, \quad \partial w/\partial n|_{\Gamma_1'} = 0, \quad w|_{\Gamma_1''} = 0, \quad \partial w/\partial n|_{\Gamma_1''} = 0$$

We will further assume that there is a one-to-one smooth mapping of the region Ω'' into Ω', such that Γ_1'' is mapped one-to-one onto Γ_1'. After appropriate replacement of the coordinates of the middle surface of the second shell we obtain that both shells are specified in the same coordinate region $\Omega = \Omega'$.

We will assume that the above mapping of the middle surface Ω'' onto the middle surface $\Omega' = \Omega$ is close to identical, so that after appropriate replacement of the variables in all the expressions and functionals for the second shell we obtain that the changes in the geometrical, elastic and force parameters are relatively small quantities. Differences between corresponding quantities will be denoted by the additional symbol Δ.

Thus, we introduce the following quantities

$$\Delta\mathbf{r}(\xi^1, \xi^2) = \mathbf{r}''(\xi^1, \xi^2) - \mathbf{r}'(\xi^1, \xi^2), \quad \Delta h = h'' - h', \quad \Delta c^{ijkl} = c''^{ijkl} - c'^{ijkl}$$

$$\Delta a^{ij} = a''^{ij} - a'^{ij}, \quad \Delta b^{ij} = b''^{ij} - b'^{ij}, \quad \Delta\Gamma^i_{jk} = \Gamma''^i_{jk} - \Gamma'^i_{jk}$$

$$\Delta J = J'' - J', \quad \Delta\mathbf{g} = \mathbf{g}'' - A_\tau\mathbf{g}'$$

The element $\mathbf{g}$ is defined by (3) while the operator $\mathbf{A}_\tau$ is introduced by the equality [1] $(A_\tau\mathbf{u}, \chi)_{H'} = (\mathbf{u}, \chi)_{H''}$. Here we have used the notation H' and H'' respectively for the energy spaces for each of the shells.

It is obvious that the shell S' has no particular advantages over the shell S''. Hence, all the equations and relations should also refer to the shell S''. To do this it is necessary to introduce the operator B_τ: $(B_\tau\mathbf{u}, \chi)_{H''} = (\mathbf{u}, \chi)_{H'}$.

The following assertion was proved in [1].

Lemma 2. Suppose that, for sufficiently small $\varepsilon > 0$

$$\|\Delta \mathbf{r}\|_{C^{(2)}(\Omega)} \leqslant \varepsilon, \quad \|\Delta h\|_{C(\Omega)} \leqslant \varepsilon, \quad \|\Delta c^{ijkl}\|_{C(\Omega)} \leqslant \varepsilon \tag{5}$$

A certain constant $m > 0$ then exists such that

$$\|\Delta a_{ij}\|_{C^{(1)}(\Omega)} \leqslant m\varepsilon, \quad \|\Delta a^{ij}\|_{C^{(1)}(\Omega)} \leqslant m\varepsilon, \quad \|\Delta b_{ij}\|_{C(\Omega)} \leqslant m\varepsilon, \quad \|\Delta b^{ij}\|_{C(\Omega)} \leqslant m\varepsilon$$

$$\|\Delta \Gamma^i_{jk}\|_{C(\Omega)} \leqslant m\varepsilon, \quad \|\Delta J\|_{C(\Omega)} \leqslant m\varepsilon, \quad 1 - m\varepsilon \leqslant \|A_\tau\| \leqslant 1 + m\varepsilon$$

$$1 - m\varepsilon \leqslant \|B_\tau\| \leqslant 1 + m\varepsilon$$

As previously [1], we will further investigate how the values of the operator of the problem depend on the variation of the problem parameters. Since formally, after changing to the new system of coordinates for the second shell, we obtain a problem in the same region and with the same type of boundary conditions and, consequently, we formally have the same problem of well-posedness as in [1], we can immediately formulate the final result.

Theorem 1. Suppose that for the shell S' there is a non-singular generalized solution of the problem of equilibrium under a load $\mathbf{g}'$, described by a functional of the work of the external forces. Suppose further that there is a shell S'' under a load $\mathbf{g}''$, and that condition (5) is satisfied and, in addition, $\| \Delta \mathbf{g} \|_{H'} \leqslant \varepsilon$. In this case, for sufficiently small ε, a generalized non-singular solution of the problem of the equilibrium of the shell S'' exists in the form $\mathbf{u}' + \Delta\mathbf{u}$, and $\| \Delta \mathbf{u} \|_{H'} \leqslant \delta(\varepsilon)$, where $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0$. Further, in a sphere of radius $\delta(\varepsilon)$ with center $\mathbf{u}$, there is exactly one generalized solution of the equilibrium problem for each of the shells.

The proof of the theorem is completely similar to the proof of Theorem 31.3 in [1]; the basis of the proof is the structure of the operator $\mathbf{G}(\mathbf{u})$ mentioned in [1].

Note that, as a rule, the singular solutions of the problems are isolated. It does not follow from Theorem 1 that, from the existence of a singular solution for one of the shells, a singular solution exists for the other shell, but it follows that, when a singular solution exists for the second shell, it necessarily lies in a certain small neighbourhood of the first singular solution.

Above we considered the case when only one part of the boundary Γ_1 is perturbed. If there are several such parts of the boundary, then, for the most part, it is impossible to map the region Ω'' into Ω' so that all the corresponding parts of the boundary are mapped one into the other. In this case it is necessary to consider a sequence of boundary-value problems, each of which differs from the previous type of boundary conditions are specified. Then, by arguing for each of the corresponding pairs of problems in this chain, we obtain that a theorem of the type of Theorem 1 also holds in this case.

The problem of the well-posedness of the problem when a highly elastic support is specified on the boundary of the region was considered in detail in [1]. If, for two shells, the form of the elastically supported contour is slightly displaced or one of them differs slightly from the other, all the above discussion can easily be transferred to this case also.

Finally, we note that strict boundary conditions with respect to the tangential components of the displacement vector are determined solely by the theorem of solvability, proved in [1], and not by the technique for proving Theorem 1. Assuming the existence of a generalized non-singular isolated solution of the problem of the equilibrium of a shell with boundary conditions of any type, we obtain a theorem of the type of Theorem 1 by the above discussions in this case also.

We will make one further observation. The continuity of the dependence of the non-singular solutions on a change in the form of the shell and the form of the boundaries and boundary conditions enables us to consider the problem of the convergence of the finite element method when the boundary of the region is not a polygon.

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Translated by R.C.G.